

THE AUTOMORPHISM GROUPS OF QUASI-GALOIS CLOSED ARITHMETIC SCHEMES

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ABSTRACT. Assume that X and Y are arithmetic schemes, i.e., integral schemes of finite types over $\text{Spec}(\mathbb{Z})$. Then X is said to be quasi-galois closed over Y if X has a unique conjugate over Y in some certain algebraically closed field, where the conjugate of X over Y is defined in an evident manner. Now suppose that $\phi : X \rightarrow Y$ is a surjective morphism of finite type such that X is quasi-galois closed over Y . In this paper the main theorem says that the function field $k(X)$ is canonically a Galois extension of $k(Y)$ and the automorphism group $\text{Aut}(X/Y)$ is isomorphic to the Galois group $\text{Gal}(k(X)/k(Y))$; in particular, ϕ must be affine. Moreover, let $\dim X = \dim Y$. Then X is a pseudo-galois cover of Y in the sense of Suslin-Voevodsky.

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INTRODUCTION

Let k_1 be an algebraic extension of a field k and let X be an algebraic variety defined over k_1 . Then by a k -automorphism σ of \bar{k} , we

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get a conjugate X^σ of X over k . X is said to be *normally algebraic* over k , defined by Weil, if X coincides with each of the conjugates of X over k (see [20]). It is well-known that algebraic varieties and their conjugates behave like conjugates of fields and have almost all of the algebraic properties (for example, see [7, 20]). But their complex topological properties are very different from each other (for example, see [13, 17]). On the other hand, in the geometric version of class field theory, following Weil's [21] and [22], Lang uses algebraic varieties to describe unramified class fields over function fields in several variables (see [12]); then in virtue of Bloch's [5] and others' foundations, Kato and Saito use algebraic fundamental groups to obtain unramified class field theory (see [10, 16]). Here, the main feature is to use the abelianized fundamental groups of algebraic or arithmetic schemes to describe abelian class fields (for example, see [11, 15, 23, 24]).

Motivated by those works, in this paper we will suggest a definition that an arithmetic variety is said to be *quasi-galois closed* if it has a unique conjugate in an algebraically closed field (see *Definition 1.1*), which can be regarded as a generalization from the notion that an algebraic variety is normally algebraic over a number field to the one that an arithmetic variety is quasi-galois closed over a fixed arithmetic one. Here, an *arithmetic variety* is an integral scheme of finite type over $\text{Spec}(\mathbb{Z})$. Then we will try to use these relevant data of such arithmetic varieties to obtain some information of Galois extensions of function fields in several variables.

The following is the *Main Theorem* of the paper (i.e., *Theorem 2.1*).

Main Theorem 0.1. (Theorem 2.1) *Let X and Y be two arithmetic varieties. Assume that X is quasi-galois closed over Y by a surjective morphism ϕ of finite type. Then there are the following statements.*

- f is affine.
- $k(X)$ is canonically a Galois extension of $k(Y)$.
- There is a group isomorphism

$$\text{Aut}(X/Y) \cong \text{Gal}(k(X)/k(Y)).$$

- Particularly, let $\dim X = \dim Y$. Then X is a pseudo-galois cover of Y in the sense of Suslin-Voevodsky.

See [18] for the definition of *pseudo-galois* covers of schemes. Note that here $k(X)$ is not necessarily algebraic over $k(Y)$ by ϕ in the first property above. That is, the morphism ϕ is not necessarily finite.

Hence, the *Main Theorem* of the paper shows us some evidence that there exists a nice relationship between quasi-galois closed arithmetic varieties and Galois extensions of functions fields in several variables.

For the case that ϕ is finite, it can be seen that quasi-galois closed arithmetic varieties behave like Galois extensions of number fields and their automorphism groups can be regarded as the Galois groups of the field extensions.

In deed, one has been attempted to use the data of such varieties X/Y to describe a given Galois extension E/F for a long time and one says that X/Y are a *model* for E/F if the Galois group $Gal(E/F)$ is isomorphic to the automorphism group $Aut(X/Y)$ (for example, see [7, 14, 15, 18, 19]). The *Main Theorem* gives us such a model for function fields in several variables.

In [18, 19], Suslin and Voevodsky obtain several good properties for pseudo-galois covers of varieties for the case that the morphism ϕ is finite, where they also give the existence of pseudo-galois covers. If the arithmetic varieties are of the same dimensions, it is seen that there is no essential difference between our “quasi-galois closed” and “pseudo-galois cover”.

However, there is a main difference between the two types of covers if the structure morphism is not finite. For example, let t be a variable over \mathbb{Q} . Then $Spec(\mathbb{Z}[t])/Spec(\mathbb{Z})$ is quasi-galois closed but not pseudo-galois. Hence, to some degree, the *Main Theorem* of the paper gives us a sufficient condition for the existence of such a pseudo-galois cover in a more generalized case in the category of arithmetic varieties, where the function fields are in several variables.

The *Main Theorem* of the paper can be regarded as a generalization of *Proposition 1.1* in [7], *Page 106*, for the case of function fields in several variables.

Now let us give some applications of quasi-galois closed covers such as the following.

In [2] we will prove the existence of quasi-galois closed covers of arithmetic schemes and then by these covers we will give an explicit construction of the geometric model for a prescribed Galois extension of a function field in several variables over a number field.

In [4] we will use quasi-galois closed covers to define and compute a qc fundamental group for an arithmetic scheme. Then we will prove that the étale fundamental group of an arithmetic scheme is a normal subgroup in our qc fundamental group. Hence, our group gives us a prior estimate of the étale fundamental group. The quotient group reflects the topological properties of the arithmetic scheme.

Particularly, in [3] we will use quasi-galois closed covers to give the computation of the étale fundamental group of an arithmetic scheme.

0.1. Outline of the Proof for the Main Theorem. The whole of §3 will be devoted to the proof of the Main Theorem of the present paper, where we will proceed in several subsections.

In §3.1 we will recall some preliminary facts on affine structures on arithmetic schemes (see [1]). Here, affine structures on a scheme behave like differential structures on a differential manifold. In §3.2 we will prove that a quasi-galois closed arithmetic variety has one and only one maximal affine structure among others with values in a fixed algebraically closed field (see *Proposition 3.9*).

In §3.3 we will define conjugations of a given field and a quasi-galois extension of a field in an evident manner. For the case of algebraic extensions, “conjugation” is exactly “conjugate” and “quasi-galois” is exactly “normal”.

Let K be a finitely generated extension of a fixed field k . In §3.4 we will demonstrate that K is quasi-galois over k if and only if K has only one conjugation over k (see *Corollary 3.14*). Moreover, K is a Galois extension of k if K is quasi-galois and separably generated over k (see the proof of *Theorem 3.26*).

Then conjugations and quasi-galois extensions for fields will be geometrically realized in arithmetic varieties. In §3.5 we will define conjugations of an open subset in an arithmetic variety in an evident manner. An open subset of an arithmetic variety is said to have a quasi-galois set of conjugations if all of its conjugations can be affinely realized in the variety.

Now let $\phi : X \rightarrow Y$ be a surjective morphism of finite type between arithmetic varieties. Suppose that X is quasi-galois closed over Y by the structure morphism ϕ .

In §3.6 we will establish a relationship between the conjugations of fields and the conjugations of open subsets in arithmetic varieties. In deed, the discussions on fields and schemes are parallel. It will be proved that affine open sets in X have quasi-galois sets of conjugations (see *Theorem 3.23*) and that the function field $k(X)$ is canonically a quasi-galois extension of the function field $k(Y)$ (see *Theorem 3.24*).

In §3.7 we will prove that the automorphism group of X over Y is isomorphic the Galois group of the function field $k(X)$ over $k(Y)$ (see *Theorem 3.26*), which is the dominant part of the Main Theorem in the paper. Finally in §3.8 we will complete the proof for the Main Theorem of the paper.

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1. NOTATION AND DEFINITIONS

1.1. Convention. In this paper, an **arithmetic variety** is an integral scheme of finite type over $\text{Spec}(\mathbb{Z})$. A k -**variety** is an integral scheme of finite type over a field k . By a **variety**, we will understand an arithmetic variety or a k -variety. Let $k(X) \triangleq \mathcal{O}_{X,\xi}$ denote the function field of a variety X (with generic point ξ).

Let X and Y be varieties over a fixed variety Z . Y is said to be a **conjugate** of X over Z if there is an isomorphism $\sigma : X \rightarrow Y$ over Z . Let $\text{Aut}(X/Z)$ denote the group of automorphisms of X over Z .

Let D be an integral domain. Denote by $\text{Fr}(D)$ the field of fractions on D . If D is a subring of a field Ω , $\text{Fr}(D)$ will be assumed to be contained in Ω .

Let E be an extension of a field F . Note that here E is not necessarily algebraic over F . Recall that E is a **Galois extension** of F if F is the invariant subfield of the Galois group $\text{Gal}(E/F)$.

1.2. Affine covering with values in a given field. Let (X, \mathcal{O}_X) be a scheme. As usual, an affine covering of the scheme (X, \mathcal{O}_X) is a family $\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ such that for each $\alpha \in \Delta$, ϕ_α is an isomorphism from an open set U_α of X onto the spectrum $\text{Spec} A_\alpha$ of a commutative ring A_α . Each $(U_\alpha, \phi_\alpha; A_\alpha) \in \mathcal{C}_X$ is called a **local chart**. An affine covering \mathcal{C}_X of (X, \mathcal{O}_X) is said to be **reduced** if $U_\alpha \neq U_\beta$ holds for any $\alpha \neq \beta$ in Δ .

Sometimes, we will denote by $(X, \mathcal{O}_X; \mathcal{C}_X)$ a scheme (X, \mathcal{O}_X) with a given affine covering \mathcal{C}_X . For the sake of brevity, a local chart $(U_\alpha, \phi_\alpha; A_\alpha)$ will be denoted by U_α or (U_α, ϕ_α) .

Let **Comm** be the category of commutative rings with identity. Fixed a subcategory **Comm**₀ of **Comm**. An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of (X, \mathcal{O}_X) is said to be **with values in Comm**₀ if $\mathcal{O}_X(U_\alpha) = A_\alpha$ holds and A_α is contained in **Comm**₀ for each $\alpha \in \Delta$.

In particular, let Ω be a field (large enough) and let **Comm**(Ω) be the category consisting of the subrings of Ω and their isomorphisms. An affine covering \mathcal{C}_X of (X, \mathcal{O}_X) with values in **Comm**(Ω) is said to be **with values in the field Ω** .

1.3. Quasi-galois closed varieties. Let X and Y be two varieties and let $f : X \rightarrow Y$ be a surjective morphism of finite type.

Definition 1.1. The variety X is said to be **quasi-galois closed** over Y by f if there is an algebraic closed field Ω and a reduced affine

covering \mathcal{C}_X of X with values in Ω such that for any conjugate Z of X over Y the following conditions are satisfied:

- (i) $(X, \mathcal{O}_X) = (Z, \mathcal{O}_Z)$ holds if (Z, \mathcal{O}_Z) has a reduced affine coverings with values in Ω .
- (ii) Each local chart contained in \mathcal{C}_Z is contained in \mathcal{C}_X for any reduced affine covering \mathcal{C}_Z of (Z, \mathcal{O}_Z) with values in Ω .

In particular, if Y is $\text{Spec}(\mathbb{Z})$ or $\text{Spec}(k)$, such a variety X is said to be a **quasi-galois closed variety**.

Remark 1.2. The existence of quasi-galois closed varieties.

- (i) For the case of varieties, the finite group actions on varieties can produce quasi-galois closed varieties (For example, see [6, 7, 14, 18, 19]).
- (ii) For the case of schemes, there is another way to obtain quasi-galois closed schemes. Let X be a scheme with a finite number of conjugates. Then the disjoint union of the conjugates of X will be quasi-galois closed over X .
- (iii) For a general case, in [2] we will prove the existence of quasi-galois closed schemes over arithmetic schemes.

2. STATEMENT OF THE MAIN THEOREM

Here is the *Main Theorem* of the present paper, which will be proved in §3.

Theorem 2.1. (Main Theorem). *Let X and Y be two arithmetic varieties. Assume that X is quasi-galois closed over Y by a surjective morphism ϕ of finite type. Then there are the following statements.*

- f is affine.
- $k(X)$ is canonically a Galois extension of $k(Y)$.
- There is a group isomorphism

$$\text{Aut}(X/Y) \cong \text{Gal}(k(X)/k(Y)).$$

- Particularly, let $\dim X = \dim Y$. Then X is a pseudo-galois cover of Y in the sense of Suslin-Voevodsky.

Remark 2.2. By the first property in *Theorem 2.1* it is seen that there exists a nice relationship between quasi-galois closed arithmetic varieties and Galois extensions of functions fields in several variables.

Remark 2.3. If $\dim X = \dim Y$, it is seen that quasi-galois closed arithmetic varieties behave like Galois extensions of number fields and their automorphism groups can be regarded as the Galois groups of the field extensions. If $\dim X > \dim Y$, *Theorem 2.1* can be regarded as a generalization of that in *Proposition 1.1* in [7], *Page 106* for function fields in several variables.

Remark 2.4. We have attempted to use the data of such varieties X/Y to describe a given finite Galois extension E/F in such a manner that X/Y are said to be a *model* for E/F if the Galois group $Gal(E/F)$ is isomorphic to the automorphism group $Aut(X/Y)$ (for example, see [7, 14, 15, 18, 19]). Hence, *Theorem 2.1* afford us such a model for function fields in several variables.

Remark 2.5. If $\dim X = \dim Y$, we have pseudo-galois covers of arithmetic varieties in the sense of Suslin-Voevodsky (see [18, 19]); it is seen that there is no essential difference between our “quasi-galois closed” and “pseudo-galois cover”. However, suppose $\dim X > \dim Y$. Then it is seen that there is a main difference between the two types of covers. For example, $Spec(\mathbb{Z}[t])/Spec(\mathbb{Z})$ is quasi-galois closed but not pseudo-galois, where t is a variable over \mathbb{Q} . Hence, *Theorem 2.1* gives us a sufficient condition for the existence of such a pseudo-galois cover in a more generalized case in the category of arithmetic varieties, where the function fields are in several variables.

3. PROOF OF THE MAIN THEOREM

In this section we will proceed in several subsections to prove the main theorem of the paper.

3.1. Affine structures. Let us recall some preliminary results on affine structures (see [?]) which will be used in the following subsections. Here, affine structures on a schemes can be regarded as a counterpart of differential structures on a manifold in topology (for example, see [8]).

Let \mathbf{Comm} be the category of commutative rings with identity. Fixed a subcategory \mathbf{Comm}_0 of \mathbf{Comm} .

Definition 3.1. A pseudogroup Γ of affine transformations (with values in \mathbf{Comm}_0) is a subcategory of \mathbf{Comm}_0 such that the algebra isomorphisms contained in Γ satisfying the conditions (i) – (v):

(i) Each $\sigma \in \Gamma$ is an isomorphism between algebras $dom(\sigma)$ and $rang(\sigma)$ contained in \mathbf{Comm}_0 , called the **domain** and **range** of σ , respectively.

(ii) Let $\sigma \in \Gamma$. Then the inverse σ^{-1} is contained in Γ .

(iii) The identity map id_A on A is contained in Γ if there is some $\delta \in \Gamma$ with $dom(\delta) = A$.

(iv) Let $\sigma \in \Gamma$. Then the isomorphism induced by σ defined on the localization $dom(\sigma)_f$ of the algebra $dom(\sigma)$ at any nonzero $f \in dom(\sigma)$ is contained in Γ .

(v) Let $\sigma, \delta \in \Gamma$. Assume for some $\tau \in \Gamma$ there are isomorphisms $\text{dom}(\tau) \cong \text{dom}(\sigma)_f$ and $\text{dom}(\tau) \cong \text{rang}(\delta)_g$ with $0 \neq f \in \text{dom}(\sigma)$ and $0 \neq g \in \text{rang}(\delta)$. Then the isomorphism factorized by $\text{dom}(\tau)$ from $\text{dom}(\sigma)_f$ onto $\text{rang}(\delta)_g$ is contained in Γ .

Let X be a topological space and let Γ be a pseudogroup of affine transformations with values in \mathbf{Comm}_0 .

Definition 3.2. An affine Γ -atlas \mathcal{A} on X (with values in \mathbf{Comm}_0) is a collection of triples $(U_j, \phi_j; A_j)$ with $j \in \Delta$, called **local charts**, satisfying the conditions (i) – (iii):

(i) For every $(U_j, \phi_j; A_j) \in \mathcal{A}$, U_j is an open subset of X and ϕ_j is an homeomorphism of U_j onto $\text{Spec}(A_j)$ with $A_j \in \Gamma$ such that $U_i \neq U_j$ holds for any $i \neq j$ in Δ .

For the sake of brevity, such a triple $(U_j, \phi_j; A_j)$ will be denoted sometimes by U_j or by a pair (U_j, ϕ_j) .

(ii) $\bigcup_{j \in \Delta} U_j$ is an open covering of X .

(iii) Take any $(U_i, \phi_i, A_i), (U_j, \phi_j, A_j) \in \mathcal{A}$ with $U_i \cap U_j \neq \emptyset$. Then there is a local chart $(W_{ij}, \phi_{ij}) \in \mathcal{A}$ with $W_{ij} \subseteq U_i \cap U_j$ such that the isomorphism between the localizations $(A_j)_{f_j}$ and $(A_i)_{f_i}$ induced by the map $\phi_j \circ \phi_i^{-1}|_{W_{ij}}: \phi_i(W_{ij}) \rightarrow \phi_j(W_{ij})$ is contained in Γ , where $\phi_i(W_{ij}) \cong \text{Spec}(A_i)_{f_i}$ and $\phi_j(W_{ij}) \cong \text{Spec}(A_j)_{f_j}$ are homeomorphic for some $f_i \in A_i$ and $f_j \in A_j$.

Definition 3.3. Two affine Γ -atlases \mathcal{A} and \mathcal{A}' on X are said to be Γ -compatible if the condition below is satisfied:

Take any $(U, \phi, A) \in \mathcal{A}$ and $(U', \phi', A') \in \mathcal{A}'$ with $U \cap U' \neq \emptyset$. Then there is a local chart $(W, \phi'') \in \mathcal{A} \cap \mathcal{A}'$ with $W \subseteq U \cap U'$ such that the isomorphism between the localizations A_f and $(A')_{f'}$ induced by the map $\phi' \circ \phi^{-1}|_W: \phi(W) \rightarrow \phi'(W)$ is contained in Γ , where $\phi(W) \cong \text{Spec} A_f$ and $\phi'(W) \cong \text{Spec}(A')_{f'}$ are homeomorphic for some $f \in A$ and $f' \in A'$.

By an **affine Γ -structure** on X (with values in \mathbf{Comm}_0) we understand a maximal affine Γ -atlas $\mathcal{A}(\Gamma)$ on X . Here, an affine Γ -atlas \mathcal{A} on X is said to be **maximal** (or **complete**) if it can not be contained properly in any other affine Γ -atlas of X .

Remark 3.4. Fixed a pseudogroup Γ of affine transformations. By Zorn's Lemma it is seen that for any given affine Γ -atlas \mathcal{A} on X there is a unique affine Γ -structure \mathcal{A}_m on X satisfying

- (i) $\mathcal{A} \subseteq \mathcal{A}_m$;
- (ii) \mathcal{A} and \mathcal{A}_m are Γ -compatible.

In such a case, \mathcal{A} is said to be a **base** for \mathcal{A}_m and \mathcal{A}_m is the affine Γ -structure defined by \mathcal{A} .

Definition 3.5. Let $\mathcal{A}(\Gamma)$ be an affine Γ -structure on X . Assume that there is a sheaf \mathcal{F} of rings on X such that (X, \mathcal{F}) is a locally ringed space and that $\phi_{\alpha*}\mathcal{F}|_{U_{\alpha}}(\text{Spec}A_{\alpha}) = A_{\alpha}$ holds for each $(U_{\alpha}, \phi_{\alpha}; A_{\alpha}) \in \mathcal{A}(\Gamma)$.

Then $\mathcal{A}(\Gamma)$ is said to be **admissible** on X and \mathcal{F} is said to be an **extension** of $\mathcal{A}(\Gamma)$.

It is evident that such a sheaf \mathcal{F} on X affords us a scheme (X, \mathcal{F}) . That is, an extension of an affine structure on a space is a scheme.

Let (X, \mathcal{O}_X) be a scheme and U_{α} an affine open set of X . Take an isomorphism $(\phi_{\alpha}, \phi_{\alpha}^{\#}) : (U_{\alpha}, \mathcal{O}_{X|U_{\alpha}}) \rightarrow (\text{Spec}A_{\alpha}, \mathcal{O}_{\text{Spec}A_{\alpha}})$. In general, the ring $\mathcal{O}_X(U)$ is isomorphic to A_{α} by $\phi_{\alpha}^{\#}$. Here, we choose the ring A_{α} to be such that $\mathcal{F}(U) = A_{\alpha}$ in the definition above. This can be done according to the preliminary facts on affine schemes (see [6]).

Remark 3.6. It is easily seen that all extensions of a fixed admissible affine structure on a space are isomorphic schemes (see [?]).

3.2. A quasi-galois closed variety has only one maximal affine structure among others with values in a fixed field. Let \mathbf{Comm}/k be the category of finitely generated algebras (with identities) over a given field k . We will consider the pseudogroup of affine transformations with values in \mathbf{Comm}/k in this subsection.

Fixed a k -variety (X, \mathcal{O}_X) with a given reduced affine covering \mathcal{C}_X . That is, each reduced affine covering gives us a pseudogroup of affine transformations in a natural manner.

In deed, define $\Gamma(\mathcal{C}_X)$ to be the set of identities $1_{A_{\alpha}} : A_{\alpha} \rightarrow A_{\alpha}$ and isomorphisms $\sigma_{\alpha\beta} : (A_{\alpha})_{f_{\alpha}} \rightarrow (A_{\beta})_{f_{\beta}}$ of k -algebras for any $0 \neq f_{\alpha} \in A_{\alpha}$ and $0 \neq f_{\beta} \in A_{\beta}$, where A_{α} and A_{β} are contained in \mathbf{Comm}/k such that there are some affine open subsets U_{α} and U_{β} of X with $(U_{\alpha}, \phi_{\alpha}), (U_{\beta}, \phi_{\beta}) \in \mathcal{C}_X$ satisfying

$$\phi_{\alpha}(U_{\alpha}) = \text{Spec}A_{\alpha}, \phi_{\beta}(U_{\beta}) = \text{Spec}A_{\beta}.$$

Then $\Gamma(\mathcal{C}_X)$ is a pseudogroup in \mathbf{Comm}/k , called the **(maximal) pseudogroup of affine transformations** in $(X, \mathcal{O}_X; \mathcal{C}_X)$.

It is seen that \mathcal{C}_X is an affine $\Gamma(\mathcal{C}_X)$ -atlas on X . Denote by $\mathcal{A}(\mathcal{C}_X)$ the affine $\Gamma(\mathcal{C}_X)$ -structure on X defined by an affine $\Gamma(\mathcal{C}_X)$ -atlas \mathcal{C}_X on X .

Remark 3.7. Let \mathbf{Comm}/\mathbb{Z} be the category of finitely generated algebras (with identities) over \mathbb{Z} . We can similarly define a pseudogroup Γ of affine transformations with values in \mathbf{Comm}/\mathbb{Z} and then discuss affine Γ -structures such as the above.

In particular, let Ω be a field and let $\mathbf{Comm}(\Omega)$ be the category consisting of the subrings of Ω and their isomorphisms. An affine structure of a variety X with values in $\mathbf{Comm}(\Omega)$ is said to be **with values in the field Ω** .

Remark 3.8. Let (X, \mathcal{O}_X) be a variety (i.e, an arithmetic variety or a k -variety).

(i) Different reduced affine coverings \mathcal{C}_X have different pseudogroups $\Gamma(\mathcal{C}_X)$ of affine transformations.

(ii) Each reduced affine covering \mathcal{C}_X is an admissible affine atlas. In particular, \mathcal{O}_X is an extension of \mathcal{C}_X on the underlying space X .

(iii) As an admissible affine atlas, each reduced affine covering \mathcal{C}_X can have many extensions \mathcal{F}_X on the space X . With such an extension we have a scheme (X, \mathcal{F}_X) , called an **associate scheme** of (X, \mathcal{O}_X) .

(iv) By *Remark 3.6* it is seen that all associate schemes of (X, \mathcal{O}_X) are isomorphic.

Proposition 3.9. Let X and Y be varieties with X quasi-galois closed over Y . Then the variety X has one and only one affine structure $\mathcal{A}(\mathcal{O}_X)$ with values in an algebraic closed field Ω satisfying the below properties:

(i) The structure sheaf \mathcal{O}_X is an extension of $\mathcal{A}(\mathcal{O}_X)$.

(ii) By set inclusion, $\mathcal{A}(\mathcal{O}_X)$ is maximal among the whole of the affine structures on the underlying space of X with values in Ω . That is, take any affine Γ -structure \mathcal{B} on the space of X with values in Ω . Then we must have $\mathcal{B} \subseteq \mathcal{A}(\mathcal{O}_X)$ and $\Gamma \subseteq \Gamma(\mathcal{A}(\mathcal{O}_X))$, where $\Gamma(\mathcal{A}(\mathcal{O}_X))$ is the maximal pseudogroup of affine transformations of $(X, \mathcal{O}_X; \mathcal{A}(\mathcal{O}_X))$.

In particular, we can choose Ω to be an fixed algebraic closure of the function field $k(X)$ of X .

Here, $\mathcal{A}(\mathcal{O}_X)$ will be called the **natural affine structure** of (X, \mathcal{O}_X) with values in Ω .

Proof. Let \mathcal{C}_X and \mathcal{C}'_X be two affine structures on the underlying space X with respect to pseudogroups Γ and Γ' respectively, which are both with values in some field Ω . As an affine structure is a maximal affine atlas, it is clear that \mathcal{C}_X and \mathcal{C}'_X are reduced affine coverings on the space X . By *Definition 1.1*, we must have either

$$\mathcal{C}_X \subseteq \mathcal{C}'_X, \Gamma \subseteq \Gamma'$$

or

$$\mathcal{C}_X \supseteq \mathcal{C}'_X, \Gamma \supseteq \Gamma'.$$

Let Σ be the set of affine structures on the underlying space X with values in Ω . By set inclusion, Σ is a partially ordered set since any

two affine structures are compatible with the pseudogroups of affine transformations.

Hence, Σ is totally ordered. The unique maximal element in Σ is the desired affine structure, where we choose the field Ω to be an algebraic closure of the functional field $\mathcal{O}_{X,\xi} = k(X)$ of X . \square

3.3. Definition for conjugations of a given field. Let K be an extension of a field k . Here K/k is not necessarily algebraic. K is said to be k -**quasi-galois** (or, **quasi-galois** over k) if each irreducible polynomial $f(X) \in F[X]$ that has a root in K factors completely in $K[X]$ into linear factors for any intermediate field $k \subseteq F \subseteq K$.

Let E be a finitely generated extension of k . The elements

$$w_1, w_2, \dots, w_n \in E \setminus k$$

are said to be a (r, n) -**nice k -basis** of E (or simply, a **nice k -basis**) if the following conditions are satisfied:

$$E = k(w_1, w_2, \dots, w_n);$$

w_1, w_2, \dots, w_r constitute a transcendental basis of E over k ;

$w_{r+1}, w_{r+2}, \dots, w_n$ are linearly independent over $k(w_1, w_2, \dots, w_r)$, where $0 \leq r \leq n$.

Definition 3.10. Let E and F be two finitely generated extensions of a field k . F is said to be a k -**conjugation** of E (or, a **conjugation** of E over k) if there is a (r, n) -nice k -basis w_1, w_2, \dots, w_n of E such that F is a conjugate of E over $k(w_1, w_2, \dots, w_r)$.

We will denote by $\tau_{(r,n)}$ such an isomorphism from F onto E over $k(w_1, w_2, \dots, w_r)$ with respect to the (r, n) -nice k -basis.

Remark 3.11. Let F be a k -conjugation of E . Then F is contained in the algebraic closure \overline{E} of E .

It will be proved that a finitely generated field is quasi-galois if and only if it has only one conjugation (see *Corollary 3.14*). For the case of algebraic extensions, this is exactly to say that a field is normal if and only if it has only one conjugate field.

3.4. A quasi-galois field has only one conjugation. We give the below criterion for a quasi-galois field by conjugations, which behaves like a normal field and its conjugate field for the case of algebraic extensions.

Theorem 3.12. *Let K be a finitely generated extension of a field k . The following statements are equivalent.*

(i) K is a quasi-galois field over k .

(ii) Take any $x \in K$ and any subfield $k \subseteq F \subseteq K$. Then each conjugation of $F(x)$ over F is contained in K .

(iii) *Each k -conjugation of K is contained in K .*

Proof. (i) \implies (ii). Take any $x \in K$ and any $k \subseteq F \subseteq K$. If x is a variable over F , the field $F(x)$ is the unique k -conjugation of $F(x)$ in $\overline{F(x)}$ ($\subseteq \overline{K}$). If x is algebraic over F , a F -conjugation of $F(x)$ which is exactly a F -conjugate of $F(x)$ is contained in K by the assumption that K is k -quasi-galois; then all F -conjugates of $F(x)$ in $\overline{F(x)}$ ($\subseteq \overline{K}$) is contained in K .

(ii) \implies (i). Let F be a field with $k \subseteq F \subseteq K$. Take any irreducible polynomial $f(X)$ over F . Suppose that $x \in K$ satisfies the equation $f(x) = 0$. Then such an F -conjugation of $F(x)$ is an F -conjugate. By (ii) it is seen that every F -conjugate $z \in \overline{F}$ of x is contained in K ; hence, K is quasi-galois over k .

(ii) \implies (iii). Hypothesize that there is a k -conjugation H of K in \overline{K} is not contained in K , that is, $H \setminus K$ is a nonempty set. Take any $x_0 \in H \setminus K$.

Choose a (r, n) -nice k -basis w_1, w_2, \dots, w_n of K which make H be a k -conjugation of K . By *Remark 3.11* it is seen that H is contained in the algebraic closure of $k(w_1, w_2, \dots, w_n)$. As w_1, w_2, \dots, w_r are all variables over k , it is seen that w_1, w_2, \dots, w_r are all contained in the intersection of H and K . By *Definition 3.10* it is seen that there is an isomorphism $\sigma : H \rightarrow K$ of fields over $k(w_1, w_2, \dots, w_r)$.

It is evident that the specified element x_0 must be algebraic over $k(w_1, w_2, \dots, w_r)$. Then the field $k(w_1, w_2, \dots, w_r, x_0)$ is a conjugate of the field $k(w_1, w_2, \dots, w_r, \sigma(x_0))$ over $k(w_1, w_2, \dots, w_r)$.

From (ii) we have $k(w_1, w_2, \dots, w_r, x_0) \subseteq K$. In particular, x_0 is contained in K , which is in contradiction with the hypothesis above. Therefore, every k -conjugation of K is in K .

(iii) \implies (ii). Take any $x \in K$ and any field F such that $k \subseteq F \subseteq K$. If x is a variable over F , $F(x)$ is the unique F -conjugation in \overline{K} of $F(x)$ itself by *Remark 3.11* again; hence, $F(x)$ is contained in K .

Now suppose that x is algebraic over F . Let $z \in \overline{K}$ be an F -conjugate of x . If $F = K$, we have $\sigma_x = id_K$; then $z = x \in K$. If $F \neq K$, from *Lemma 3.13* below we have a field $F(z, v_1, v_2, \dots, v_s, w_{s+1}, \dots, w_m)$ that is an F -conjugation of K ; it is seen that the element z is contained in an F -conjugation of K ; as $k \subseteq K$, an F -conjugation of K must be an k -conjugation of K ; by (iii) we must have $z \in K$. This proves (ii). \square

Lemma 3.13. *Fixed a finitely generated extension K of a field k and a field F with $k \subseteq F \subsetneq K$. Let $x \in K$ be algebraic over F and let z be a conjugate of x over F . Then there is a (s, m) -nice $F(x)$ -basis*

v_1, v_2, \dots, v_m of K and an F -isomorphism τ from the field

$$K = F(x, v_1, v_2, \dots, v_s, v_{s+1}, \dots, v_m)$$

onto a field of the form

$$F(z, v_1, v_2, \dots, v_s, w_{s+1}, \dots, w_m)$$

such that

$$\tau(x) = z, \tau(v_1) = v_1, \dots, \tau(v_s) = v_s$$

where $w_{s+1}, w_{s+2}, \dots, w_m$ are elements contained in an extension of F . In particular, we have

$$w_{s+1} = v_{s+1}, w_{s+2} = v_{s+2}, \dots, w_m = v_m$$

if z is not contained in $F(v_1, v_2, \dots, v_m)$.

Proof. We will proceed in two steps according to the assumption that $s = 0$ or $s \neq 0$.

Step 1. Let $s \neq 0$. That is, v_1 is a variable over $F(x)$.

Let σ_x be the F -isomorphism between fields $F(x)$ and $F(z)$ with $\sigma_x(x) = z$. From the isomorphism σ_x we obtain an isomorphism σ_1 of $F(x, v_1)$ onto $F(z, v_1)$ defined by

$$\sigma_1 : \frac{f(v_1)}{g(v_1)} \mapsto \frac{\sigma_x(f)(v_1)}{\sigma_x(g)(v_1)}$$

for any polynomials $f[X_1], g[X_1] \in F(x)[X_1]$ with $g[X_1] \neq 0$.

It is easily seen that $g(v_1) = 0$ if and only if $\sigma_x(g)(v_1) = 0$. Hence, the map σ_1 is well-defined.

Similarly, for the elements $v_1, v_2, \dots, v_s \in K$ that are variables over $F(x)$, there is an isomorphism

$$\sigma_s : F(x, v_1, v_2, \dots, v_s) \longrightarrow F(z, v_1, v_2, \dots, v_s)$$

of fields defined by

$$x \longmapsto z \text{ and } v_i \longmapsto v_i$$

for $1 \leq i \leq s$, where we have the restrictions

$$\sigma_{i+1}|_{F(x, v_1, v_2, \dots, v_i)} = \sigma_i.$$

If $s = m$, we have $K = F(x, v_1, v_2, \dots, v_s)$ and it follows that the field $F(z, v_1, v_2, \dots, v_s)$ is an F -conjugation of K . We put $s \leq m - 1$.

Step 2. Let $s = 0$. That is exactly to consider the case $v_{s+1} \in K$ since v_{s+1} is algebraic over the field $F(v_1, v_2, \dots, v_s) \subseteq K$. We have two cases for the element v_{s+1} .

Case (i). Suppose that z is not contained in $F(v_1, v_2, \dots, v_{s+1})$.

We have an isomorphism σ_{s+1} between the fields $F(x, v_1, v_2, \dots, v_{s+1})$ and $F(z, v_1, v_2, \dots, v_{s+1})$ given by

$$x \longmapsto z \text{ and } v_i \longmapsto v_i$$

with $1 \leq i \leq s+1$.

The map σ_{s+1} is well-defined. In deed, by the below **Claim**[†] it is seen that $f(v_{s+1}) = 0$ holds if and only if $\sigma_s(f)(v_{s+1}) = 0$ holds for any polynomial $f(X_{s+1}) \in F(x, v_1, v_2, \dots, v_s)[X_{s+1}]$.

Case (ii). Suppose that z is contained in the field $F(v_1, v_2, \dots, v_{s+1})$.

By the below **Claim**^{††} we have an element v'_{s+1} contained in an extension of F such that the fields $F(x, v_{s+1})$ and $F(z, v'_{s+1})$ are isomorphic over F .

Then by the same procedure as in *Case (i)* of **Claim**[†] it is seen that the fields $F(x, v_{s+1}, v_1, v_2, \dots, v_s)$ and $F(z, v'_{s+1}, v_1, v_2, \dots, v_s)$ are isomorphic over F .

Hence, in such a manner we have an F -isomorphism τ from the field

$$F(x, v_1, v_2, \dots, v_s, v_{s+1}, \dots, v_m)$$

onto the field of the form

$$F(z, v_1, v_2, \dots, v_s, w_{s+1}, \dots, w_m)$$

such that

$$\tau(x) = z, \tau(v_1) = v_1, \dots, \tau(v_s) = v_s,$$

where $w_{s+1}, w_{s+2}, \dots, w_m$ are elements contained in an extension of F . This completes the proof of the lemma. \square

Claim[†]. Given any $f(X, X_1, X_2, \dots, X_{s+1})$ in the polynomial ring $F[X, X_1, X_2, \dots, X_{s+1}]$. Suppose that z is not contained in the field $F(v_1, v_2, \dots, v_{s+1})$. Then $f(x, v_1, v_2, \dots, v_{s+1}) = 0$ holds if and only if $f(z, v_1, v_2, \dots, v_{s+1}) = 0$ holds.

Proof. Here we use Weil's algebraic theory of specializations (See [20]) to prove the claim. For v_{s+1} there are two cases:

$$v_{s+1} \in \overline{F};$$

$$v_{s+1} \in \overline{F(v_1, v_2, \dots, v_{s+1})} \setminus \overline{F},$$

where \overline{F} denotes the algebraic closure of the field F .

Case (i). Let $v_{s+1} \in \overline{F(v_1, v_2, \dots, v_{s+1})} \setminus \overline{F}$.

By *Theorem 1* in [20], *Page 28*, it is clear that (z) is a (generic) specialization of (x) over F since z and x are conjugates over F . From *Proposition 1* in [20], *Page 3*, it is seen that $F(v_1, v_2, \dots, v_{s+1})$ and the field $F(x)$ are free with respect to each other over F since x is algebraic over F . That is, $F(v_1, v_2, \dots, v_{s+1})$ is a free field over F

with respect to (x) . By *Proposition 3* in [20], *Page 4*, it is seen that $F(v_1, v_2, \dots, v_{s+1})$ and the algebraic closure \overline{F} are linearly disjoint over F . That is, $F(v_1, v_2, \dots, v_{s+1})$ is a regular extension of F (For detail, see [20], *Page 18*).

Then by *Theorem 5* in [20], *Page 29*, it is seen that $(z, v_1, v_2, \dots, v_{s+1})$ is a (generic) specialization of $(x, v_1, v_2, \dots, v_{s+1})$ over F since (z) is a (generic) specialization of (x) over F and $(v_1, v_2, \dots, v_{s+1})$ is a (generic) specialization of $(v_1, v_2, \dots, v_{s+1})$ itself over F .

Case (ii). Let $v_{s+1} \in \overline{F}$.

By the above assumption for z it is seen that z is not contained in the field $F(v_{s+1})$. It is easily seen that there is an isomorphism between the fields $F(x, v_{s+1})$ and $F(z, v_{s+1})$. It follows that (z, v_{s+1}) is a (generic) specialization of (x, v_{s+1}) over F . By the same procedure as in the above *Case (i)* it is seen that $(z, v_{s+1}, v_1, v_2, \dots, v_s)$ is a (generic) specialization of $(x, v_{s+1}, v_1, v_2, \dots, v_s)$ over F .

Now take any polynomial $f(X, X_1, X_2, \dots, X_{s+1})$ over F . According to *Cases (i)-(ii)*, it is seen that $f(x, v_1, v_2, \dots, v_{s+1}) = 0$ holds if and only if $f(z, v_1, v_2, \dots, v_{s+1}) = 0$ holds by the theory for generic specializations. This completes the proof. \square

Claim^{††}. Assume that $F(u)$ and $F(u')$ are isomorphic over F given by $u \mapsto u'$. Let w be an element contained in an extension of F . Then there is an element w' contained in some extension of F such that the fields $F(u, w)$ and $F(u', w')$ are isomorphic over F .

Proof. It is immediate from *Proposition 4* in [20], *Page 30*. \square

Corollary 3.14. Let K be a finitely generated extension of a field k . Then K is a quasi-galois field over k if and only if K has one and only one conjugation over k .

Proof. Prove \Leftarrow . Let K have only one k -conjugation H . We must have $H = K$ and then each k -conjugation of K is contained in K . By *Theorem 3.12* it is seen that K is a quasi-galois field over k .

Prove \Rightarrow . Let K be a k -quasi-galois field and H a k -conjugation of K . Choose a k -isomorphism τ of H onto K and a (s, m) -nice k -basis v_1, v_2, \dots, v_m of K such that H is a conjugate of K over F by τ , where $F \triangleq k(v_1, v_2, \dots, v_s)$. We have $F \subseteq H \subseteq K$.

Hypothesize $H \subsetneq K$. Fixed any $x_0 \in K \setminus H$. For the element x_0 there are two cases.

Case (i). Let x_0 be a variable over H . We have

$$\dim_k H = \dim_k K = s < \infty$$

since H and K are conjugations over k . But from $x_0 \in K \setminus H$, it is seen that

$$1 + \dim_k H = \dim_k H(x_0) \leq \dim_k K$$

hold; from it we will obtain a contradiction.

Case (ii). Let x_0 be algebraic over H . As $\overline{H} \subseteq \overline{F}$, we have $x_0 \in \overline{F}$; it follows that x_0 is algebraic over F . It is clear that we have

$$[H : F] = [K : F] < \infty$$

since H is a conjugate of K over F by τ . But from $x_0 \in K \setminus H$, it is seen that

$$2 + [H : F] \leq [H(x_0) : F] \leq [K : F]$$

hold; from it we will obtain a contradiction.

Therefore, the set $K \setminus H$ is empty and we must have $K = H$. \square

3.5. Definition for conjugations of an open set. The notion on conjugations of an open set in a given variety that will be defined in this subsection can be regarded as a geometric counterpart to that for the case of fields in §3.3.

Let us first consider the case for integral domains. Here we let $Fr(D)$ denote the fractional field of an integral domain D .

Definition 3.15. Let $D \subseteq D_1 \cap D_2$ be three integral domains.

(i) The ring D_1 is said to be D -**quasi-galois** (or, **quasi-galois** over D) if the field $Fr(D_1)$ is a quasi-galois extension of $Fr(D)$.

(ii) Assume that there is a (r, n) -nice k -basis w_1, w_2, \dots, w_n of the field $Fr(D_1)$ and an F -isomorphism $\tau_{(r,n)} : Fr(D_1) \rightarrow Fr(D_2)$ of fields such that $\tau_{(r,n)}(D_1) = D_2$, where $k = Fr(D)$ and we set $F \triangleq k(w_1, w_2, \dots, w_r)$ to be contained in the intersection $Fr(D_1) \cap Fr(D_2)$.

Then the ring D_1 is said to be a D -**conjugation** of the ring D_2 (or, a **conjugation** of D_2 over D).

Then consider an integral scheme Z . Let $z \in Z$. By the structure sheaf \mathcal{O}_Z on Z , we have the canonical embeddings

$$i_U^Z : \mathcal{O}_Z(U) \rightarrow k(Z);$$

$$i_z^Z : \mathcal{O}_{Z,z} \rightarrow k(Z);$$

$$i_U^z : \mathcal{O}_Z(U) \rightarrow \mathcal{O}_{Z,z}$$

for every open set U of Z containing z , where $k(Z) = \mathcal{O}_{Z,\xi}$ is the function field of X and ξ is the generic point of Z .

We will identify these integral domains with their images, that is, we will take the rings

$$\mathcal{O}_Z(U) \subseteq \mathcal{O}_{Z,z} \subseteq k(Z)$$

as subrings of the function field $k(Z)$. This leads us to obtain the following definitions.

Now fixed any two k -varieties (or, arithmetic varieties) X and Y and let $\phi : X \rightarrow Y$ be a morphism of finite type. Take a point $y \in \phi(X)$ and an open set V in Y with $V \cap \phi(X) \neq \emptyset$.

Definition 3.16. Assume that U_1 and U_2 are open sets of X such that either U_1 or U_2 is contained in $\phi^{-1}(V)$. The open set U_1 is said to be a V -**conjugation** of the open set U_2 if the ring $i_{U_1}^X(\mathcal{O}_X(U_1))$ ($\subseteq k(X)$) is a conjugation of the ring $i_{U_2}^X(\mathcal{O}_X(U_2))$ ($\subseteq k(X)$) over the ring $i_{\phi^{-1}(V)}^X(\phi^\#(\mathcal{O}_Y(V)))$ ($\subseteq k(X)$), where $\phi^\# : \mathcal{O}_Y(V) \rightarrow \phi_*\mathcal{O}_X(V) = \mathcal{O}_X(\phi^{-1}(V))$ is the ring homomorphism.

If U_1 and U_2 are both contained in $\phi^{-1}(V)$, such a V -conjugation is said to be **geometric**.

Remark 3.17. It is seen that the above conjugation of an open set is well-defined since we have

$$\begin{aligned} & i_{\phi^{-1}(V)}^X(\phi^\#(\mathcal{O}_Y(V))) \\ &= i_{U_1}^X(i_{\phi^{-1}(V)}^{U_1}(\phi^\#(\mathcal{O}_Y(V)))) \\ &= i_{U_1 \cap U_2}^X(i_{\phi^{-1}(V)}^{U_1 \cap U_2}(\phi^\#(\mathcal{O}_Y(V)))) \\ &= i_{U_2}^X(i_{\phi^{-1}(V)}^{U_2}(\phi^\#(\mathcal{O}_Y(V)))). \end{aligned}$$

In particular, if ϕ is surjective, we have

$$\begin{aligned} & \phi^\#(k(Y)) \subseteq k(X); \\ & \phi^\#(i_V^Y(\mathcal{O}_Y(V))) = i_{\phi^{-1}(V)}^X(\phi^\#(\mathcal{O}_Y(V))). \end{aligned}$$

Definition 3.18. Assume that either $x_1 \in X$ or $x_2 \in X$ is contained in $\phi^{-1}(y)$. The point x_1 is said to be a y -**conjugation** of the point x_2 if the ring $i_{x_1}^X(\mathcal{O}_{X,x_1})$ ($\subseteq k(X)$) is a conjugation of the ring $i_{x_2}^X(\mathcal{O}_{X,x_2})$ ($\subseteq k(X)$) over the ring $i_{x_1}^X(\phi^\#(\mathcal{O}_{Y,y}))$ ($\subseteq k(X)$), where $\phi^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x_1}$ is the ring homomorphism.

If x_1 and x_2 are both contained in $\phi^{-1}(y)$, such a y -conjugation is said to be **geometric**.

Remark 3.19. The above conjugation of a point is well-defined. In deed, by *Remark 3.17* we have

$$i_{x_1}^X(\phi^\#(\mathcal{O}_{Y,y})) = i_{x_2}^X(\phi^\#(\mathcal{O}_{Y,y}))$$

as subrings of $k(X)$ according to the preliminary facts on direct systems of rings.

Let A be a commutative ring with identity. A is said to be **affinely realized** in X by an open set U of X if we have $A = \mathcal{O}_X(U)$. A is said

to be **affinely realized** in X by a point x of X if we have $A = \mathcal{O}_{X,x}$. This is a hint of the following notion for the case of varieties.

Definition 3.20. An open set $U \subseteq \phi^{-1}(V)$ in the variety X is said to have a **quasi-galois set of V -conjugations** in X if each conjugation A of the ring $i_U^X(\mathcal{O}_X(U))$ over the ring $i_{\phi^{-1}(V)}^X(\phi^\#(\mathcal{O}_Y(V)))$ can be affinely realized canonically by an open set U_A of X such that $A = i_{U_A}^X(\mathcal{O}_X(U_A))$.

It is easily seen that such an open set U_A can be contained in the set $\phi^{-1}(V)$.

Definition 3.21. A point $x \in \phi^{-1}(y)$ in the variety X is said to have a **quasi-galois set of y -conjugations** in X if each conjugation A of the ring $i_x^X(\mathcal{O}_{X,x})$ over the ring $i_x^X(\phi^\#(\mathcal{O}_{Y,y}))$ can be affinely realized canonically by a point x_A of X such that $A = i_{x_A}^X(\mathcal{O}_{X,x_A})$.

In particular, the fiber $\phi^{-1}(y)$ is said to be **quasi-galois** over y if each point of the fiber $\phi^{-1}(y)$ has a quasi-galois set of y -conjugations in X .

Remark 3.22. Let $y \in V$. By *Theorem 3.23* below it is easily seen that each point $x_0 \in \phi^{-1}(y)$ has a quasi-galois set of y -conjugations implies that each affine open set $U \subseteq \phi^{-1}(V)$ containing x_0 has a quasi-galois set of V -conjugations in X .

3.6. Quasi-galois closed varieties and conjugations of open sets.

In this subsection we will obtain some properties of quasi-galois closed varieties by virtue of conjugations of open sets.

Theorem 3.23. *Let X and Y be two k -varieties (or, two arithmetic varieties) such that X is quasi-galois closed over Y by a surjective morphism ϕ of finite type.*

(i) *Fixed any affine open set V of Y . Then each affine open set $U \subseteq \phi^{-1}(V)$ has a quasi-galois set of V -conjugations in X .*

(ii) *Let Ω be an fixed algebraic closure of the functional field $k(X)$. Then we have*

$$\mathcal{O}_X(U) \subseteq \mathcal{O}_{X,x_0} \subseteq \Omega$$

exactly as subsets for any point $x \in X$ and any affine open set U of X containing x .

Proof. Let Ω be an fixed algebraic closure of the functional field $k(X)$ of X . By *Proposition 3.9* we have the natural affine structure $\mathcal{A}(\mathcal{O}_X)$ of the variety (X, \mathcal{O}_X) such that $\mathcal{A}(\mathcal{O}_X)$ is with values in Ω and that \mathcal{O}_X is an extension of $\mathcal{A}(\mathcal{O}_X)$.

(i) Hypothesize that there is an affine open set $U_0 \subseteq \phi^{-1}(V)$ such that a conjugation H of $i_{U_0}^X(\mathcal{O}_X(U_0))$ over $i_{\phi^{-1}(V)}^X(\phi^\#(\mathcal{O}_Y(V)))$ can not be affinely realized canonically by any open set U' of X with $H = i_{U'}^X(\mathcal{O}_X(U'))$.

Evidently, $H \neq i_{U_0}^X(\mathcal{O}_X(U_0))$. From the field Ω we have $\mathcal{O}_X(U_0) = i_{U_0}^X(\mathcal{O}_X(U_0))$ and then $H \neq \mathcal{O}_X(U_0)$.

Put

$$\mathcal{C}'_X = \{(U_0, \phi'_0; H)\} \bigcup (\mathcal{A}(\mathcal{O}_X) \setminus \{(U_0, \phi_0; A_0)\})$$

where $(U_0, \phi_0; A_0) \in \mathcal{A}(\mathcal{O}_X)$ and $\phi'_0(U_0) = \text{Spec}(H)$ is an isomorphism.

Let $\Gamma(\mathcal{C}'_X)$ be the maximal pseudogroup of affine transformations in $(X, \mathcal{O}_X; \mathcal{C}'_X)$ and let $\mathcal{A}'(\mathcal{O}_X)$ be the affine $\Gamma(\mathcal{C}'_X)$ -structure defined by the reduced affine covering \mathcal{C}'_X .

By gluing schemes (see [9]), it is easily seen that $\mathcal{A}'(\mathcal{O}_X)$ is admissible and there is a sheaf \mathcal{O}'_X on X such that \mathcal{O}'_X is an extension of $\mathcal{A}'(\mathcal{O}_X)$. Then (X, \mathcal{O}'_X) is a scheme such that $\mathcal{O}'_X(U_0)$ is exactly equal to the ring H since they are both subrings of Ω .

As $\mathcal{A}(\mathcal{O}_X)$ and $\mathcal{A}'(\mathcal{O}_X)$ are both with values in Ω , in virtue of (ii) of *Proposition 3.9* we have $\mathcal{A}(\mathcal{O}_X) \supseteq \mathcal{A}'(\mathcal{O}_X)$; as affine structures are reduced coverings of X , we must have

$$\mathcal{O}_X(U_0) = \mathcal{O}'_X(U_0) = H$$

since $(U_0, \phi'_0; H) \in \mathcal{A}'(\mathcal{O}_X)$, which will be in contradiction with the hypothesis above. Therefore, each affine open set $U \subseteq \phi^{-1}(V)$ has a quasi-galois set of V -conjugations in X .

(ii) Fixed a point $x_0 \in X$. Let I_{x_0} (respectively, J_{x_0}) be the index family of open sets (respectively, affine open sets) of X containing x_0 . By set inclusion, I_{x_0} and J_{x_0} are partially ordered sets and then are directed sets. It is easily seen that J_{x_0} and I_{x_0} are cofinal since affine open sets for a base for the topology on the space of X . Hence, the stalk \mathcal{O}_{X, x_0} at x_0 is the direct limit of the system of rings $\mathcal{O}_X(U)$ with $U \in J_{x_0}$.

Now consider the natural affine structure $\mathcal{A}(\mathcal{O}_X)$ with values in Ω . Take each local chart $(U, \phi; A) \in \mathcal{A}(\mathcal{O}_X)$ with $U \in J_{x_0}$. It is seen that we have

$$i_U^X(\mathcal{O}_X(U)) = \mathcal{O}_X(U) = A \subseteq \Omega$$

since \mathcal{O}_X is an extension of $\mathcal{A}(\mathcal{O}_X)$.

Similarly, take any affine open sets U_1, U_2 of X containing x_0 such that $U_1 \subseteq U_2$. We have

$$i_{U_1}^{U_2}(\mathcal{O}_X(U_2)) = \mathcal{O}_X(U_2) \subseteq \Omega.$$

It follows that for the stalk of \mathcal{O}_X at x_0 we have

$$\mathcal{O}_{X,x_0} = \bigcup_{U \in J_{x_0}} \mathcal{O}_X(U) \subseteq \Omega.$$

(Please notice that all isomorphisms i_U^X and $i_{U_1}^{U_2}$ here are exactly identity maps only for affine open sets!) \square

Theorem 3.24. *Let X and Y be two k -varieties (or, two arithmetic varieties) such that X is quasi-galois closed over Y by a surjective morphism ϕ of finite type. Then the function field $k(X)$ is a quasi-galois extension over the image $\phi^\sharp(k(Y))$ of the function field $f(Y)$.*

Proof. By Proposition 3.9 it is seen that there is the natural affine structure $\mathcal{A}(\mathcal{O}_X)$ of the variety (X, \mathcal{O}_X) with values in Ω and \mathcal{O}_X is an extension of $\mathcal{A}(\mathcal{O}_X)$, where Ω is an fixed algebraic closure of the functional field $k(X)$ of X .

Now fixed any conjugation H of $k(X)$ over $\phi^\sharp(k(Y))$. Take any element $w_0 \in H$. Let $\sigma : H \rightarrow k(X)$ be an isomorphism over $\phi^\sharp(k(Y))$. Put $u_0 = \sigma(w_0)$.

In virtue of Theorem 3.23 we have

$$\mathcal{O}_X(U) \subseteq k(X) = \mathcal{O}_{X,\xi} \subseteq \Omega$$

exactly as subsets of Ω for any affine open set U of X , where ξ is the generic point of X . Then we have

$$\bigcup_U \mathcal{O}_X(U) = \mathcal{O}_{X,\xi}$$

since affine open sets U of X form a base for the topology of X .

It follows that there are affine open subsets V_0 of Y and $U_0 \subseteq \phi^{-1}(V_0)$ of X such that u_0 is contained in $\mathcal{O}_X(U_0)$. By Theorem 3.23 again it is seen that there is some affine open set W_0 of X such that W_0 is a V -conjugation of U_0 and that the element w_0 is contained in $\mathcal{O}_X(W_0)$.

Hence, w_0 is contained in $k(X) = \mathcal{O}_{X,\xi} \subseteq \Omega$. This proves any given conjugation H of $k(X)$ over $\phi^\sharp(k(Y))$ is contained in $k(X)$. From Theorem 3.12 it is seen that the function field $k(X)$ is a quasi-galois extension of the field $\phi^\sharp(k(Y))$. \square

At last we have the following corollary.

Corollary 3.25. *Let X and Y be two k -varieties (or, two arithmetic varieties) such that X is quasi-galois closed over Y by a surjective morphism ϕ of finite type. Suppose that each V -conjugation of U is geometric for any affine open sets $V \subseteq Y$ and $U \subseteq \phi^{-1}(V)$.*

Then each point $x_0 \in \phi^{-1}(y_0)$ has a quasi-galois set of geometric y -conjugations in X for any point $y_0 \in Y$.

Proof. Fixed a point $y_0 \in Y$ and a point $x_0 \in \phi^{-1}(y_0)$. Take any affine open sets $V \subseteq Y$ and $U \subseteq \phi^{-1}(V)$ such that $x_0 \in U$ and $y_0 \in V$. By *Theorem 3.23* it is seen that U has a quasi-galois set of geometric V -conjugations; from *Theorem 3.24* it is seen that each point $x_0 \in \phi^{-1}(y_0)$ has a quasi-galois set of geometric y -conjugations in X for any point $y_0 \in Y$. \square

3.7. Automorphism groups of quasi-galois closed varieties and Galois groups of the function fields. For automorphism groups of quasi-galois closed varieties, we have the following result.

Theorem 3.26. *Let X and Y be two k -varieties (or, two arithmetic varieties) such that X is quasi-galois closed over Y by a surjective morphism ϕ of finite type. Suppose that $k(X)/\phi^\sharp(k(Y))$ is separably generated. Then the function field $k(X)$ is a Galois extension of $\phi^\sharp(k(Y))$ and there is a group isomorphism*

$$\text{Aut}(X/Y) \cong \text{Gal}(k(X)/\phi^\sharp(k(Y))).$$

Proof. (i). Prove that the function field $k(X)$ is a finitely generated Galois extension of $\phi^\sharp(k(Y))$.

Without loss of generality, assume that $k(X)$ is a transcendental extension over $\phi^\sharp(k(Y))$. By *Corollary 3.14* and *Theorem 3.24* it is seen that every conjugation of $k(X)$ over $\phi^\sharp(k(Y))$ is $k(X)$ itself. It needs to prove that there exists a $\sigma_0 \in \text{Gal}(k(X)/\phi^\sharp(k(Y)))$ such that $\phi^\sharp(k(Y))$ is the invariant subfield of σ_0 .

In deed, take a (r, n) -nice F -basis v_1, v_2, \dots, v_n of $k(X)$. By the assumption above it is seen that $r \geq 1$ holds and $k(X)$ is an algebraic Galois extension of the field $F_0 \triangleq \phi^\sharp(k(Y))(v_1, v_2, \dots, v_r)$. Fixed any $\tau_0 \in \text{Gal}(k(X)/F_0)$ with $\tau_0 \neq \text{id}_{k(X)}$. Let $\tau_1 \in \text{Gal}(F_0/\phi^\sharp(k(Y)))$ be given by

$$v_1 \mapsto \frac{1}{v_1}, v_2 \mapsto \frac{1}{v_2}, \dots, v_r \mapsto \frac{1}{v_r}.$$

We have a $\sigma_0 \in \text{Gal}(k(X)/\phi^\sharp(k(Y)))$ defined by τ_0 and τ_1 in such a manner

$$\begin{aligned} & \frac{f(v_1, v_2, \dots, v_n)}{g(v_1, v_2, \dots, v_n)} \in k(X) \\ & \mapsto \frac{f(\tau_1(v_1), \tau_1(v_2), \dots, \tau_1(v_r), \tau_0(v_{r+1}), \dots, \tau_0(v_n))}{g(\tau_1(v_1), \tau_1(v_2), \dots, \tau_1(v_r), \tau_0(v_{r+1}), \dots, \tau_0(v_n))} \in k(X) \end{aligned}$$

for any polynomials $f(X_1, X_2, \dots, X_n)$ and $g(X_1, X_2, \dots, X_n) \neq 0$ over the field $\phi^\sharp(k(Y))$ such that $g(v_1, v_2, \dots, v_n) \neq 0$. By τ_0 it is seen that we have

$$g(v_1, v_2, \dots, v_n) = 0$$

if and only if

$$g(\tau_1(v_1), \tau_1(v_2), \dots, \tau_1(v_r), \tau_0(v_{r+1}), \dots, \tau_0(v_n)) = 0$$

holds. Hence, σ_0 is well-defined.

It is seen that $\phi^\sharp(k(Y))$ is the invariant subfield of σ_0 and then $\phi^\sharp(k(Y))$ is the invariant subfield of $\text{Gal}(k(X)/\phi^\sharp(k(Y)))$. Therefore, $k(X)$ is a Galois extension of $\phi^\sharp(k(Y))$.

(ii). Now let $\mathcal{A}(\mathcal{O}_X)$ be the natural affine structure of the variety (X, \mathcal{O}_X) with values in an fixed algebraic closure Ω of the functional field $k(X)$.

For an open set H in X , we have an isomorphism $\tau_H : \Gamma(H, \mathcal{O}_X) \cong \mathcal{O}_X(H)$ of algebras and an embedding $i_H^X : \mathcal{O}_X(H) \rightarrow \mathcal{O}_{X,\xi} \subseteq \Omega$, where ξ is the generic point of x .

For the sake of convenience, all such rings $\Gamma(H, \mathcal{O}_X)$ and $\mathcal{O}_X(H)$ are regarded as the subrings of the function field $k(X)$ by the maps $i_H^X \circ \tau_H$ and i_H^X , respectively.

The function field $k(X) = \mathcal{O}_{X,\xi}$ is regarded as the set of the elements of the forms (U, f) , where U is an open set of X and f is an element of $\mathcal{O}_X(U)$ ($\subseteq \Omega$). That is,

$$k(X) = \{(U, f) : f \in \mathcal{O}_X(U) \text{ and } U \subseteq X \text{ is open}\}.$$

In the following we will proceed in several steps to demonstrate that there exists an isomorphism

$$t : \text{Aut}(X/Y) \cong \text{Gal}(k(X)/\phi^\sharp(k(Y)))$$

of groups.

Step 1. Fixed any automorphism $\sigma = (\sigma, \sigma^\sharp) \in \text{Aut}(X/Y)$. That is, $\sigma : X \rightarrow X$ is a homeomorphism and $\sigma^\sharp : \mathcal{O}_X \rightarrow \sigma_*\mathcal{O}_X$ is an isomorphism of sheaves of rings on X . As $\dim X < \infty$, we have $\sigma(\xi) = \xi$. It follows that

$$\sigma^\sharp : k(X) = \mathcal{O}_{X,\xi} \rightarrow \sigma_*\mathcal{O}_{X,\xi} = k(X)$$

is an automorphism of $k(X)$. Let $\sigma^{\sharp-1}$ denote the inverse of σ^\sharp .

Take any open subset U of X . We have the restriction

$$\sigma = (\sigma, \sigma^\sharp) : (U, \mathcal{O}_X|_U) \rightarrow (\sigma(U), \mathcal{O}_X|_{\sigma(U)})$$

of open subschemes. That is,

$$\sigma^\sharp : \mathcal{O}_X|_{\sigma(U)} \rightarrow \sigma_*\mathcal{O}_X|_U$$

is an isomorphism of sheaves on $\sigma(U)$. In particular,

$$\sigma^\sharp : \mathcal{O}_X(\sigma(U)) = \mathcal{O}_X|_{\sigma(U)}(\sigma(U)) \rightarrow \mathcal{O}_X(U) = \sigma_*\mathcal{O}_X|_U(\sigma(U))$$

is an isomorphism of rings.

For every $f \in \mathcal{O}_X|_U(U)$, we have

$$f \in \sigma_* \mathcal{O}_X|_U(\sigma(U));$$

hence

$$\sigma^{\sharp-1}(f) \in \mathcal{O}_X(\sigma(U)).$$

Now define a mapping

$$t : \text{Aut}(X/Y) \longrightarrow \text{Gal}(k(X)/\phi^\sharp(k(Y)))$$

given by

$$\sigma = (\sigma, \sigma^\sharp) \longmapsto t(\sigma) = \langle \sigma, \sigma^{\sharp-1} \rangle$$

such that

$$\langle \sigma, \sigma^{\sharp-1} \rangle : (U, f) \in \mathcal{O}_X(U) \longmapsto (\sigma(U), \sigma^{\sharp-1}(f)) \in \mathcal{O}_X(\sigma(U))$$

is a mapping of $k(X)$ into $k(X)$.

Step 2. Prove that t is well-defined. In deed, given any

$$\sigma = (\sigma, \sigma^\sharp) \in \text{Aut}(X/Y).$$

For any $(U, f), (V, g) \in k(X)$, we have

$$(U, f) + (V, g) = (U \cap V, f + g)$$

and

$$(U, f) \cdot (V, g) = (U \cap V, f \cdot g);$$

then we have

$$\begin{aligned} & \langle \sigma, \sigma^{\sharp-1} \rangle ((U, f) + (V, g)) \\ &= \langle \sigma, \sigma^{\sharp-1} \rangle ((U \cap V, f + g)) \\ &= (\sigma(U \cap V), \sigma^{\sharp-1}(f + g)) \\ &= (\sigma(U \cap V), \sigma^{\sharp-1}(f)) + (\sigma(U \cap V), \sigma^{\sharp-1}(g)) \\ &= (\sigma(U), \sigma^{\sharp-1}(f)) + (\sigma(V), \sigma^{\sharp-1}(g)) \\ &= \langle \sigma, \sigma^{\sharp-1} \rangle ((U, f)) + \langle \sigma, \sigma^{\sharp-1} \rangle ((V, g)) \end{aligned}$$

and

$$\begin{aligned} & \langle \sigma, \sigma^{\sharp-1} \rangle ((U, f) \cdot (V, g)) \\ &= \langle \sigma, \sigma^{\sharp-1} \rangle ((U \cap V, f \cdot g)) \\ &= (\sigma(U \cap V), \sigma^{\sharp-1}(f \cdot g)) \\ &= (\sigma(U \cap V), \sigma^{\sharp-1}(f)) \cdot (\sigma(U \cap V), \sigma^{\sharp-1}(g)) \\ &= (\sigma(U), \sigma^{\sharp-1}(f)) \cdot (\sigma(V), \sigma^{\sharp-1}(g)) \\ &= \langle \sigma, \sigma^{\sharp-1} \rangle ((U, f)) \cdot \langle \sigma, \sigma^{\sharp-1} \rangle ((V, g)). \end{aligned}$$

It follows that $\langle \sigma, \sigma^{\sharp-1} \rangle$ is an automorphism of $k(X)$.

It needs to prove that $\langle \sigma, \sigma^{\sharp-1} \rangle$ is an isomorphism over $\phi^\sharp(k(Y))$. In deed, consider the given morphism $\phi = (\phi, \phi^\sharp) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of schemes. Evidently, $\phi(\xi)$ is the generic point of Y and ξ is invariant under any automorphism $\sigma \in \text{Aut}(X/Y)$. Then $\sigma^\sharp : \mathcal{O}_{X, \xi} \rightarrow \mathcal{O}_{X, \xi}$

is an isomorphism of algebras over $\phi^\sharp(\mathcal{O}_{Y,\phi(\xi)}) = \phi^\sharp(k(Y))$. Hence,
 $\langle \sigma, \sigma^{\sharp-1} \rangle|_{\phi^\sharp(k(Y))} = id_{\phi^\sharp(k(Y))}$.

This proves

$$\langle \sigma, \sigma^{\sharp-1} \rangle \in Gal(k(X)/\phi^\sharp(k(Y))).$$

That is, t is a well-defined map.

Prove that t is a homomorphism between groups. In fact, take any

$$\sigma = (\sigma, \sigma^\sharp), \delta = (\delta, \delta^\sharp) \in Aut(X/Y).$$

By preliminary facts on schemes (see [6]) we have

$$\delta^{\sharp-1} \circ \sigma^{\sharp-1} = (\delta \circ \sigma)^{\sharp-1};$$

then

$$\langle \delta, \delta^{\sharp-1} \rangle \circ \langle \sigma, \sigma^{\sharp-1} \rangle = \langle \delta \circ \sigma, \delta^{\sharp-1} \circ \sigma^{\sharp-1} \rangle.$$

Hence, the map

$$t : Aut(X/Y) \rightarrow Gal(k(X)/\phi^\sharp(k(Y)))$$

is a homomorphism of groups.

Step 3. Prove that t is injective. Assume $\sigma, \sigma' \in Aut(X/Y)$ such that $t(\sigma) = t(\sigma')$. We have

$$(\sigma(U), \sigma^{\sharp-1}(f)) = (\sigma'(U), \sigma'^{\sharp-1}(f))$$

for any $(U, f) \in k(X)$. In particular, we have

$$(\sigma(U_0), \sigma^{\sharp-1}(f)) = (\sigma'(U_0), \sigma'^{\sharp-1}(f))$$

for any $f \in \mathcal{O}_X(U_0)$ and any affine open subset U_0 of X such that $\sigma(U_0)$ and $\sigma'(U_0)$ are both contained in $\sigma(U) \cap \sigma'(U)$.

As \mathcal{O}_X is an extension of $\mathcal{A}(\mathcal{O}_X)$, there are three subrings

$$A_0 = \mathcal{O}_X(U_0), B_0 = \mathcal{O}_X(\sigma(U_0)), \text{ and } B'_0 = \mathcal{O}_X(\sigma'(U_0))$$

of $k(X)$ such that

$$B_0 = \sigma^{\sharp-1}(A_0) = \sigma'^{\sharp-1}(A_0) = B'_0.$$

By preliminary facts on affine schemes (see [6]) again, it is seen that

$$\sigma|_{U_0} = \sigma'|_{U_0}$$

holds as isomorphisms of schemes. As U_0 is dense in X , we have

$$\sigma = \sigma|_{\overline{U_0}} = \sigma'|_{\overline{U_0}} = \sigma'$$

on the whole of X . This proves that t is an injection.

Step 4. Prove that t is surjective. Fixed any element ρ of the group $Gal(k(X)/\phi^\sharp(k(Y)))$.

As $k(X) = \{(U_f, f) : f \in \mathcal{O}_X(U_f) \text{ and } U_f \subseteq X \text{ is open}\}$, we have

$$\rho : (U_f, f) \in k(X) \longmapsto (U_{\rho(f)}, \rho(f)) \in k(X),$$

where U_f and $U_{\rho(f)}$ are open sets in X , f is contained in $\mathcal{O}_X(U_f)$, and $\rho(f)$ is contained in $\mathcal{O}_X(U_{\rho(f)})$.

We will proceed in the following several sub-steps to prove that each element of $\text{Gal}(k(X)/\phi^\sharp(k(Y)))$ give us a unique element of $\text{Aut}(X/Y)$.

(a) Fixed any affine open set V of Y . Prove that for each affine open set $U \subseteq \phi^{-1}(V)$ there is an affine open set U_ρ in X such that ρ determines an isomorphism between affine schemes $(U, \mathcal{O}_X|_U)$ and $(U_\rho, \mathcal{O}_X|_{U_\rho})$.

In fact, take any local chart $(U, \phi; A_U) \in \mathcal{A}(\mathcal{O}_X)$ with $U \subseteq \phi^{-1}(V)$ for some affine open set V of Y . Here $\mathcal{A}(\mathcal{O}_X)$ is the natural affine structure of the variety X with values in Ω . As \mathcal{O}_X is an extension of $\mathcal{A}(\mathcal{O}_X)$, by *Theorem 3.23* we have

$$A = \mathcal{O}_X(U) = \{(U_f, f) \in k(X) : U_f \supseteq U\}$$

since U is an affine open set of X . Put

$$B = \{(U_{\rho(f)}, \rho(f)) \in k(X) : (U_f, f) \in A\}.$$

Then B is a subring of $k(X)$. As ρ is an isomorphism over $\phi^\sharp(k(Y))$, it is seen that by ρ the rings A and B are isomorphic algebras over $\phi^\sharp(k(Y))$. It follows that A and B are conjugations over $\phi^\sharp(\mathcal{O}_Y(V))$.

By *Theorem 3.23* again it is seen that U has a quasi-galois set of V -conjugations in X . Then there is an open set U_ρ that is a V -conjugation of U such that $B = \mathcal{O}_X(U_\rho)$. As U is affine open, it is clear that U_ρ is affine open.

Hence, by ρ we have a unique isomorphism

$$\lambda_U = \left(\lambda_U, \lambda_U^\sharp \right) : (U, \mathcal{O}_X|_U) \rightarrow (U_\rho, \mathcal{O}_X|_{U_\rho})$$

of the affine open subscheme in X such that

$$\rho|_{\mathcal{O}_X(U)} = \lambda_U^{\sharp-1} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(U_\rho).$$

(b) Take any affine open sets $V \subseteq Y$ and $U, U' \subseteq \phi^{-1}(V)$. Prove that

$$\lambda_U|_{U \cap U'} = \lambda_{U'}|_{U \cap U'}$$

holds as morphisms of schemes.

In fact, by the above construction for each λ_U it is seen that λ_U^\sharp and $\lambda_{U'}^\sharp$ coincide on the intersection $U \cap U'$ since we have

$$\rho|_{\mathcal{O}_X(U \cap U')} = \lambda_U|_{U \cap U'}^{\sharp-1} : \mathcal{O}_X(U \cap U') \rightarrow \mathcal{O}_X((U \cap U')_\rho);$$

$$\rho|_{\mathcal{O}_X(U \cap U')} = \lambda_{U'}|_{U \cap U'}^{\sharp-1} : \mathcal{O}_X(U \cap U') \rightarrow \mathcal{O}_X((U \cap U')_\rho).$$

For any point $x \in U \cap U'$, we must have $\lambda_U(x) = \lambda_{U'}(x)$. Otherwise, if $\lambda_U(x) \neq \lambda_{U'}(x)$, will have an affine open subset X_0 of X that contains one of the two points $\lambda_U(x)$ and $\lambda_{U'}(x)$ but does not contain the other since the underlying space of X is a Kolmogorov space. Assume $\lambda_U(x) \in X_0$ and $\lambda_{U'}(x) \notin X_0$. We choose an affine open subset U_0 of X such that $x \in U_0 \subseteq U \cap U'$ and $\lambda_U(U_0) \subseteq X_0$ since we have

$$\lambda_U(U \cap U') = (U \cap U')_\rho \subseteq U_\rho;$$

$$\lambda_{U'}(U \cap U') = (U \cap U')_\rho \subseteq U'_\rho.$$

However, by the definition for each λ_U , we have

$$\lambda_U(U_0) = (U_0)_\rho = \lambda_{U'}(U_0);$$

then

$$\lambda_{U'}(x) \in (U_0)_\rho \subseteq X_0,$$

where there will be a contradiction. Hence, λ_U and $\lambda_{U'}$ coincide on $U \cap U'$ as mappings of spaces.

(c) By gluing λ_U along all such affine open subsets U , we have a homeomorphism λ of X onto X as a topological space given by

$$\lambda : x \in X \mapsto \lambda_U(x) \in X$$

where x belongs to U and U is an affine open subset of X such that $\phi(U)$ is contained in some affine open subset V of Y . That is, $\lambda|_U = \lambda_U$. By (b) it is seen that λ is well-defined. It is clear that λ is also an automorphism of the scheme (X, \mathcal{O}_X) .

Show that λ is contained in $\text{Aut}(X/Y)$ such that $t(\lambda) = \rho$. In deed, as ρ is an isomorphism of $k(X)$ over $\phi^\#(k(Y))$, it is seen that the isomorphism λ_U is over Y by ϕ for any affine open subset U of X ; then λ is an automorphism of X over Y by ϕ such that $t(\lambda) = \rho$ holds.

This proves that there exists $\lambda \in \text{Aut}(X/Y)$ such that $t(\lambda) = \rho$ for each $\rho \in \text{Gal}(k(X)/\phi^\#(k(Y)))$.

Hence, t is surjective. This completes the proof. \square

Corollary 3.27. Let X and Y be arithmetic varieties and let X be quasi-galois closed over Y by a surjective morphism ϕ of finite type. Then there is a natural isomorphism

$$\mathcal{O}_Y \cong \phi_*(\mathcal{O}_X)^{\text{Aut}(X/Y)}$$

where $(\mathcal{O}_X)^{\text{Aut}(X/Y)}(U)$ denotes the invariant subring of $\mathcal{O}_X(U)$ under the natural action of $\text{Aut}(X/Y)$ for any open subset U of X .

Proof. Fixed any affine open sets U_0 of X and V_0 of Y with $U_0 \subseteq \phi^{-1}(V_0)$. By *Theorem 3.26* we have

$$\phi^\#(\mathcal{O}_Y(V_0)) = (\mathcal{O}_X)^{\text{Aut}(X/Y)}(U_0) = \phi_*(\mathcal{O}_X)^{\text{Aut}(X/Y)}(V_0)$$

since $k(X) = Fr(\mathcal{O}_X(U_0))$ and $k(Y) = Fr(\mathcal{O}_X(V_0))$.

Now take any open set V of Y and put $U = \phi^{-1}(V)$. We must have

$$\phi^\sharp(\mathcal{O}_Y(V)) = (\mathcal{O}_X)^{Aut(X/Y)}(U) = \phi_*(\mathcal{O}_X)^{Aut(X/Y)}(V).$$

Otherwise, if there is some element w contained in the difference set $(\mathcal{O}_X)^{Aut(X/Y)}(U) \setminus \phi^\sharp(\mathcal{O}_Y(V))$, we will have

$$w \in \mathcal{O}_X^{Aut(X/Y)}(U_1) \setminus \phi^\sharp(\mathcal{O}_Y(V_1))$$

and then we will obtain a contradiction, where $U_1 \subseteq U$ and $V_1 \subseteq V$ are affine open sets such that $U_1 \subseteq \phi^{-1}(V_1)$. This completes the proof. \square

Remark 3.28. Let X and Y be arithmetic varieties and let X be quasi-galois closed over Y by a surjective morphism ϕ of finite type. By *Corollary 3.27* it is easily seen that the morphism f must be affine.

3.8. Proof of the main theorem. Now we are ready to prove the main theorem of the paper, *Theorem 2.1* in §2.

Proof. (Proof of Theorem 2.1.) By *Theorem 3.26* and *Remark 3.28* it needs only to prove that X is a pseudo-galois cover of Y if X and Y have the same dimensions.

Let $\dim X = \dim Y$ and $G = Aut(X/Y)$. It is clear that ϕ is invariant under the natural action of $Aut(X/Y)$ on X . By *Corollary 3.27*, we have $\mathcal{O}_Y \cong \phi_*(\mathcal{O}_X)^G$. By *Theorem 3.26* again it is seen that G is a finite group. By §5 of [18], it is immediate that ϕ is finite and then X is a pseudo-galois cover of Y by ϕ . \square

Remark 3.29. Fixed any two arithmetic varieties X and Y such that X is quasi-galois closed over Y by a surjective morphism ϕ of finite type. By *Theorem 2.1* it is seen that the natural action of automorphism group $Aut(X/Y)$ on the fiber $\phi^{-1}(y)$ is transitive at each $y \in Y$.

It follows that each point $x_0 \in \phi^{-1}(y_0)$ has a quasi-galois set of geometric y_0 -conjugations in X for any point y_0 of Y .

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THE AUTOMORPHISM GROUPS OF QUASI-GALOIS CLOSED ARITHMETIC SCHEMES **29**

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